International Journal of Theoretical Physics, Vol. 34, No. 2, 1995

Relation Between Higgs and Weinberg Angles in Minimal SUSY from Diophantine Analysis

Robert L. Pease[1,2]

Received October 3, 1994

Diophantine quantization is applied to neutral Higgs and gauge bosons in minimal SUSY, with the additional requirement that β, like θ_w, be a rational angle. One finds for the Higgs mixing angles that $\alpha = -\pi/4$ and that $\beta = \theta_w$ or $\pi/2 - \theta_w$. The *CP*-odd Higgs *A* turns out to be degenerate in mass with the *Z*, and the two neutral *CP*-even Higgs bosons have masses of about 123 and 37.5 GeV, respectively. The charged Higgs has a mass of about 121.5 GeV. Further, if $\beta > \pi/4$, the infrared quasi-fixed-point solution yields a *t*-quark mass in the neighborhood of 170 GeV, in agreement with recent data.

Diophantine quantization (Pease, 1970, 1988) involves treating mass relations as Diophantine equations and seeking solutions in integers, analogous to the sets (3, 4, 5), (5, 12, 13), etc., for the Pythagorean equation $x^2 + y^2 = z^2$. It was first applied (Pease, 1970, 1988) to the Gell-Mann–Okubo meson-mass relation (Okubo, 1962; Gell-Mann, 1961)

$$m_\pi^2 + 3m_\eta^2 = 4m_K^2 \tag{1}$$

for which the simplest nontrivial solution was the set (2, 8, 7). The procedure not only gave integers proportional to the experimental masses (Particle Data Group, 1992)

$$m_\pi = 135\text{–}140 \text{ MeV}, \qquad m_\eta = 547 \text{ MeV}, \qquad m_K = 494\text{–}498 \text{ MeV}$$

but also set a unit mass—the GCF of the three meson masses—of 70 MeV [$=(\hbar c/e^2)m_e c^2$], as originally proposed by Nambu (1952).

[1] Physics Department, State University of New York, New Paltz, New York 12561.
[2] Deceased. Submitted by the estate of the author.

0020-7748/95/0200-0183$07.50/0

There is an analog to this in the standard model (Pease, 1988). If one looks carefully at the values of the W and Z masses (Particle Data Group, 1992)

$$m_W = 80.22 \pm 0.26 \text{ GeV}$$

$$m_Z = 91.173 \pm 0.020 \text{ GeV}$$

one finds that, within experimental limits,

$$\cos \theta_w = m_W/m_Z = 15/17$$

Not only is this the ratio of two integers, but these particular two integers (15, 17) happen to be two sides of a right triangle, all three of which (8, 15, 17) can be expressed as integers. If we define a rational angle as one of the acute angles of such a triangle, then θ_w is, quite interestingly, a rational angle. A simple consequence of this, which we shall use below, is that the sine and cosine of such an angle must be rational.

We shall now apply this analysis to the scalar Higgs-gauge boson mass relation of minimal SUSY (Inoue *et al.*, 1982*a*,*b*; Flores and Sher, 1983)

$$m_H^2 + m_h^2 = m_A^2 + m_Z^2 \tag{2}$$

with the auxiliary relation

$$\begin{aligned} m_{H,h}^2 = \tfrac{1}{2}\{m_A^2 + m_Z^2 \\ \pm [(m_A^2 + m_Z^2)^2 - 4m_Z^2 m_A^2 \cos^2 2\beta]^{1/2}\} \end{aligned} \tag{3}$$

which yields a simple expression (Gunion *et al.*, 1990) for the mixing angle β:

$$|\cos 2\beta| = \frac{m_H m_h}{m_A m_Z} \tag{4}$$

The other mixing angle α can be given by (Gunion *et al.*, 1990)

$$\cos 2\alpha = -\cos 2\beta \frac{m_A^2 - m_Z^2}{m_H^2 - m_h^2} \tag{5}$$

We proceed by treating equation (2) as a Diophantine equation and also requiring that $\sin \beta$ and $\cos \beta$ be rational.

To simplify notation, we use a unit mass (Pease, 1988), the GCF of the Z and W masses (Particle Data Group, 1992)

$$M_0 = m_Z/17 = m_W/15 \cong 5.36 \text{ GeV} \tag{6}$$

Then we write

$$A \equiv m_A/M_0, \qquad H = m_H/M_0, \qquad h = m_h/M_0, \qquad W = 15, \qquad Z = 17 \tag{7}$$

which allows us to rewrite equations (2), (4), and (5), respectively, as

$$H^2 + h^2 = A^2 + 289 \tag{8}$$

$$|\cos 2\beta| = \frac{Hh}{17A} \tag{9}$$

$$\cos 2\alpha = -\cos 2\beta \left(\frac{A^2 - 289}{H^2 - h^2}\right) \tag{10}$$

Further, depending upon whether $\beta < \pi/4$ (upper sign) or $\beta > \pi/4$ (lower sign), we can write

$$\sin^2 \beta = \frac{1}{2}(1 \mp \cos 2\beta) = \frac{1}{2}\left(1 \mp \frac{Hh}{17A}\right) \tag{11}$$

or

$$\sin \beta = \left(\frac{17A \mp Hh}{34A}\right)^{1/2} \qquad \begin{matrix} (\beta < \pi/4) \\ (\beta > \pi/4) \end{matrix} \tag{12}$$

Our task now is to find integer values of A, H, and h satisfying equation (8) and also leaving expression (12) rational, which means that the terms under the radical must be the quotient of two perfect squares. The search is simplified by the requirement (Gunion *et al.*, 1990, equation 4.78) that, if $\sin \beta \neq 1$ and $h > 0$,

$$0 < h < 17 \tag{13}$$

plus, of course, any further limitations imposed by experiment.

Since the radical in equation (12) must be rational, the factor of 17 in the denominator would require either (1) a second factor of 17 in the denominator or (2) a canceling factor of 17 in the numerator. Since h cannot be a multiple of 17, assumption 2 requires that $H = 17i$, where i is an integer. Then equation (8) would become

$$289i^2 + h^2 = A^2 + 289$$

which may be written

$$(A + h)(A - h) = 289(i^2 - 1)$$

Then either both $(A + h)$ and $(A - h)$ are multiples of 17, which violates equation (13), or one of the factors is a multiple of 289, in which case—questions of rationality aside—$m_A \gtrsim 1.5$ TeV, which would be unrealistically high.

Hence we adopt assumption 1 and write

$$A = 17k \qquad (k = 1, 2, 3, \ldots) \tag{14}$$

whence equation (8) becomes

$$H^2 + h^2 = (k^2 + 1)289 \tag{15}$$

This must be checked for integer H and h by a search, and the results substituted into equation (12) to test for rationality; from equations (13) and (15), H cannot be a multiple of 17.

The search is limited by the inequality

$$17k < H < 17(k^2 + 1)^{1/2}$$

so if we require that a meaningful search include the first viable candidate $H = 17k + 1$, we find we are limited to $k \le 8$.

The search yields four sets of values:

$$k = 1:\quad A = 17,\quad H = 23,\quad h = 7 \tag{16a}$$

$$k = 2:\quad A = 34,\quad H = 38,\quad h = 1 \tag{16b}$$

$$k = 3:\quad A = 51,\quad H = 53,\quad h = 9 \tag{16c}$$

$$k = 8:\quad A = 136,\quad H = 137,\quad h = 4 \tag{16d}$$

However, only set (16a) satisfies the rationality condition. [Note that the two β's in equation (12) are complements of each other.] So we are left with the set of values

$$A = 17,\qquad H = 23,\qquad h = 7 \tag{17}$$

whence

$$\sin\beta = \begin{cases} 8/17, & \beta < \pi/4 \quad (18a) \\ 15/17, & \beta > \pi/4 \quad (18b) \end{cases}$$

We first note, from comparing equation (17) with equations (6) and (8), that

$$m_A = m_Z = 91.2 \text{ GeV} \tag{19}$$

so the CP-odd neutral Higgs scalar is degenerate in mass with the Z boson. This relation was also recently derived by Ma (1994) using another method.

From equations (6), (7), and (17) we can also get the masses of the two CP-even neutral Higgs:

$$m_H \cong 123 \text{ GeV} \tag{20a}$$

$$m_h \cong 37.5 \text{ GeV} \tag{20b}$$

We can also get the mass of the charged Higgs by using the mass relation (Inoue *et al.*, 1982*a*,*b*; Flores and Sher, 1983)

$$m_{H^\pm}^2 = m_A^2 + m_W^2 \tag{21}$$

which gives

$$m_{H^\pm} \cong 121.5 \text{ GeV} \tag{22}$$

We now consider the effects on the Higgs mixing angles.

From equations (2) and (19) we can write

$$m_H^2 + m_h^2 = 2m_Z^2 \tag{23}$$

Then, using the last of equations (A12) of Gunion *et al.* (1990), we find, remembering that $\cos 2\beta \neq 0$, that

$$\cos 2\alpha = 0 \tag{24a}$$

or

$$\alpha = -\pi/4 \tag{24b}$$

Further, most surprising of all, if we recall that the Weinberg angle θ_w is characterized by

$$\cos \theta_w = m_W/m_Z = 15/17 \tag{25}$$

we have a relation between Higgs and Weinberg angles

$$\beta = \begin{cases} \theta_w, & \beta < \pi/4 \\ \pi/2 - \theta_w, & \beta > \pi/4 \end{cases} \tag{26}$$

Finally, if $\beta > \pi/4$, $\sin \beta = 15/17$, whence the infrared quasi-fixed-point solution (Bardeen *et al.*, 1994) yields a t-quark mass

$$m_t \cong (190\text{–}210) \sin \beta \quad \text{GeV} \simeq 168\text{–}175 \text{ GeV} \tag{27}$$

within the range of recent data (CDF Collaboration, 1994).

It is quite possible that equations (24) represent some $H \leftrightarrow h$ symmetry principle. Certainly, from inspection of the couplings in Appendix A of Gunion *et al.* (1990), this interchange corresponds to $\sin \alpha \leftrightarrow \pm\cos \alpha$ in the lepton interactions, and $\cos 2\alpha$ is prominent in the quartic interactions involving H and h. Equation (26) may well represent another symmetry principle. These two equations and their consequences are being investigated further.

Our value (20b) of m_h is less than the LEP minimum[3] of 43 GeV, particularly given our rather high value of $\sin^2(\beta - \alpha)$. However, we must bear in mind that our value of m_A closes the $Z \to Ah$ channel, and that we are dealing with tree-level results, while recent LEP results are adjusted for

[3]For a good summary see, e.g., Felcini (1992).

radiative corrections. If we permit $\tan \beta < 1$, the limit can drop (Particle Data Group, 1992) to 29 GeV.

We note from comparison of equations (20a) and (22) that we are close to isotriplecy (Pease, 1991). Exact isotriplecy would not have satisfied our rationality condition; however, our values for m_H and $m_{H^\pm}$ are sufficiently close that we would not encounter the problems discussed in Pease (1991).

The rationality condition is also being investigated further, as is the question of relations between coupling constants.

REFERENCES

Bardeen, W. A., Carena, M., Pokorski, S., and Wagner, C. E. M. (1994). *Physics Letters B*, **320**, 110.

CDF Collaboration (F. Abe *et al.*). (1994). FERMILAB Pub-94/097-E CDF (April 1994).

Felcini, M. (1992). CERN-PPL/92-208 (7 December 1992).

Flores, R. A., and Sher, M. (1983). *Annals of Physics*, **148**, 95.

Gell-Mann, M. (1961). California Institute of Technology Report No. CTSL-20 (unpublished) [Reprinted in Gell-Mann, M., and Ne'eman, Y. (1964). *The Eightfold Way*, Benjamin, New York].

Gunion, J. F., Haber, H. E., Kane, G., and Dawson, S. (1990). *The Higgs Hunter's Guide*, Addison-Wesley, Reading, Massachusetts.

Inoue, K., Kakuto, A., Komatsu, H., and Takeshita, S. (1982*a*). *Progress of Theoretical Physics*, **67**, 1889.

Inoue, K., Kakuto, A., Komatsu, H., and Takeshita, S. (1982*b*). *Progress of Theoretical Physics*, **68**, 927.

Ma, E. (1994). *Physical Review D*, **49**, 1663.

Nambu, Y. (1952). *Progress of Theoretical Physics*, **7**, 595.

Okubo, S. (1962). *Progress of Theoretical Physics*, **27**, 1949.

Particle Data Group (K. Hisaka *et al.*). (1992). *Physical Review D*, **45** (11/Part II) (1 June 1992).

Pease, R. L. (1970). *Physical Review D*, **2**, 1969.

Pease, R. L. (1988). *Nuovo Cimento*, **99A**, 75.

Pease, R. L. (1991). *Physics Letters B*, **261**, 399.